MMP Learning Seminar Week 83

Tools for Effective Birationality

- singularities in bounded families
- log birational boundedness of certain pairs
- boundedness of singularities on non-klt centres

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Prop 4.2 ("Singularities in Bounded Families")

Let $\varepsilon \in \mathbb{R}^{>0}$, $\mathcal{P}$ bounded set of couples.

$\Rightarrow$ $\exists\, \delta > 0$ depending on ε, $\mathcal{P}$ satisfying ⊛

⊛
- (X, B) ε-lc proj pair
- T reduced divisor s.t. $(X, \operatorname{Supp} B + T) \in \mathcal{P}$
- $L \geq 0$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor s.t.

  $\exists\, N \sim_{\mathbb{R}} L$ where $\begin{cases} N \text{ supported on } T \\ |\text{coeffs of } N| \leq \delta \end{cases}$

$\Longrightarrow (X, B+L)$ klt.

pf. $\boxed{\text{Plan}}$ 1. Assume all members of $\mathcal{P}$ have dim d and induct on d

2. Reduce to the case where
   - $(X, \operatorname{Supp} B + T)$ log smooth
   - B, T have no common components
   - T is very ample

3. Use induction hypothesis on general member of linear systems or components of B

[Step 1] Base case $d=1$

$$\varepsilon \leqq \deg L = \deg N \leqq \underbrace{\delta \deg T}_{\text{bounded}}. \qquad (X)$$

Assume $d \geqq 2$ and assume I.H.

[Step 2-1] Restrict to log smooth case

Pick "simultaneously (!)" a log resolution $\phi : W \to X$ of $(X, B+T)$ s.t.

$$K_W + B_W = \phi^*(K_X + B) + E$$

- (W, B_W) ε-lc
- $(W, \operatorname{Supp} B_W + T_W)$ is log smooth

  & belongs to a bounded set of couple S

  ($T_W :=$ (birational transform of T) + $\operatorname{Ex}(\phi)$)

$$\begin{cases} L_W = \phi^* L \\ N_W = \phi^* N \end{cases} \Rightarrow \exists m \in \mathbb{N} \text{ s.t. } (\text{coeffs of } N_W) \leqq m\delta$$

So we can replace $\mathcal{P}$, (X, B), T, L, N, δ

with S, (W, B_W), T_W, L_W, N_W, $m\delta$

and assume that $(X, \operatorname{Supp} B + T)$ is log smooth.

[Step 2-2] Reduce to the case

$$\begin{cases} B,\ T \text{ have no common components} \\ T \text{ very ample} \end{cases} \quad \text{--- } (*)$$

$T = \sum_{i=1}^{q} T_i$ where T_i = irred components

$\Rightarrow$ $\exists$ distinct very ample prime divisors $\Lambda_1, \cdots, \Lambda_{2q}$ s.t.

$\Lambda_j \not\subseteq \mathrm{Supp}(B)$ and $T_i \sim \Lambda_{2i} - \Lambda_{2i-1}$. Let $\Lambda := \sum \Lambda_j$

- $N = \sum a_i T_i \sim N' := \sum a_i \Lambda_{2i} - \sum a_i \Lambda_{2i-1}$
- $(X, \mathrm{Supp}\, B + \Lambda)$ belongs to a bounded set $\mathcal{R}$.

$\Rightarrow$ can replace $T, N, \mathcal{P}$ with $\Lambda, N', \mathcal{R}$

and $(*)$ is satisfied.

[Step 3] $(X, B+L)$ is klt

Suppose $\exists$ D prime divisor on birational model of X

s.t. $a(D, X, B+L) \leq 0$.

<u>Case 0</u> D is an irred divisor on X

$$T^{d-1} \cdot L = T^{d-1} \cdot N \underset{\substack{\uparrow \\ (\because T \text{ very ample})}}{\leq} \delta T^d \Rightarrow (\text{coeffs of } L) \leq \delta T^d.$$

$$\Rightarrow \mu_D(B+L) \leq 1 - \varepsilon + \delta T^d < 1. \qquad (\times)$$

For the remaining cases, assume D is excep'l over X, and $V :=$ center of D on X.

Case 1 $\quad V \not\subseteq \operatorname{Supp}(B)$

In $a(D, X, L) = a(D, X, B+L) \leq 0$.

$\exists\, r \in \mathbb{N},\ H \in |rT|$: (general member)
$$\begin{cases} H \text{ irred + smooth} \\ V \cap H \neq \emptyset \\ (X, H) \text{ plt} \\ (X, H+L) \text{ not plt near any component of } V\cap H \end{cases}$$

(Inverse of Adjunction) [KM] Theorem 5.50

$\Rightarrow (H, L|_H)$ not klt near any component of $V \cap H$ — (#)

Also, (coeffs of $T|_H$) ≤ 2, so $N|_H$ is supported in $T_H := \operatorname{Supp} T|_H$ with absolute value of coeffs $\leq 2\delta$.

Finally, $(H, T_H) \in \mathcal{Q}$ for some bounded family $\mathcal{Q}$.

$\Rightarrow$ Can use induction hypothesis on $(H, 0)$, T_H, $L|_H$, $N|_H$ to define 2δ

$\Rightarrow (H, L|_H)$ klt. Contradiction with (#).

(X)

Case 2 V is inside some component S of B

$\Delta := B + (1 - \underbrace{\mu_S B}_{(=:b)}) S.$ (... so that $\mu_S \Delta = 1$)

& define Δ_S s.t. $K_S + \Delta_S = (K_X + \Delta)|_S$.

(X, Δ) plt $\Rightarrow$ (S, Δ_S) ε-lc.

Since $\underset{\text{(proved in Case 0)}}{l := \mu_S L \leq \delta T^d}$, we can assume

$l < \varepsilon \leq 1 - b \implies B + L \leq \Delta + L - lS.$

$\Rightarrow V$ is non-klt centre of $(X, \Delta + L - lS)$

$\Rightarrow S$ "

$(\because \mu_S(\Delta + L - lS) = 1)$

$\Rightarrow (X, \Delta + L - lS)$ is not plt near V

(Inverse of Adjunction) $\Rightarrow$ $(S, \Delta_S + (L - lS)|_S)$ not klt.

We can take

$T' = T +$ (extra components) so that

$\exists S' \sim S$ s.t. S' supported in T' with bounded value of coeffs.

Finally, $(S, \operatorname{Supp} \Delta_S + T'|_S) \in \mathcal{R}$ bounded set of couples.

$\Rightarrow$ use induction hypothesis on (S, Δ_S), $T'|_S$, $(L - lS')|_S$, $(N - lS')|_S$. $\Rightarrow$ Contradiction. $\boxtimes$

Prop 4.4 ("Log birational boundedness of certain pairs")

Let $d, v \in \mathbb{N}$, $\varepsilon \in \mathbb{R}^{>0}$.

$\Rightarrow$ $\exists c \in \mathbb{R}^{>0}$ and a bounded set of couples $\mathcal{P}$ depending only on d, v, ε satisfying ⊛ :

⊛
- X normal proj var with dim d
- $B \geq 0$ $\mathbb{R}$-divisor with coeffs $\in \{0\} \cup [\varepsilon, \infty)$
- $M \geq 0$ nef $\mathbb{Q}$-divisor s.t.

$$\begin{cases} |M| \text{ defines a birational map} \\ \mathrm{vol}(M) \leq v \\ M - (K_X + B) \text{ is pseudo-effective} \\ \forall D \text{ component of } M : \mu_D(B+M) \geq 1 \end{cases}$$

$\Rightarrow$ $\exists$ proj log smooth couple $(\overline{X}, \Sigma_{\overline{X}}) \in \mathcal{P}$ and a birational map $\psi : \overline{X} \dashrightarrow X$ s.t.

(1) $\mathrm{Ex}(\psi)$, (birational transform of $\mathrm{Supp}(B+M)$) $\subseteq \mathrm{Supp}\, \Sigma_{\overline{X}}$

(2) (Coeffs of $M_{\overline{X}} := \psi^* M$) $\leq c$

(3) $\exists$ resolution $W \to X$ s.t. $|A_W|$ is basepoint free ($A_W :=$ movable part of $|M_W|$) and $\forall$ common resolution of $\overline{X}$, W

$$\begin{tikzcd} & X' \arrow[dl] \arrow[d] & \\ & W \arrow[d] & \\ \overline{X} \arrow[r, dashed, "\psi"'] & X & \end{tikzcd}$$

: $A_{X'} := A_W|_{X'} \sim 0 \,/\, \overline{X}$.

pf. **Idea**

$$\begin{tikzcd}
\overline{X} \arrow[dr] & & W \arrow[dl] \arrow[d] \\
& \widetilde{X} \arrow[r, dashed] & X
\end{tikzcd}$$

(1) Since $|M|$ defines a birational map,
we can first think of taking a log resolution
$\phi : W \to X$ of $(X, \mathrm{Supp}(B+M))$ s.t.
$|A_W|$ is basepoint free & defines a birational contraction.

$$\Big(M_W := \phi^* M \sim \underset{\binom{\text{movable}}{\text{part}}}{A_W} + \underset{\binom{\text{fixed}}{\text{part}}}{R_W} \Big)$$

$W \to \widetilde{X}$ = birational contraction determined by A_W
and then take an appropriate resolution $\overline{X} \to \widetilde{X}$.

(2) To define $\Sigma_{\overline{X}}$, construct a boundary divisor on W s.t. satisfies boundedness conditions on volumes,
and then use the results in [HMX (Auto)]
to conclude boundedness of collection of couples.

[Step 1] Construct $\phi : W \to X$, M_W, A_W as in Idea (1).

We assume A_W is a general member of $|A_W|$.

(and let $A := \phi_* A_W$)

[Step 2] Construction of boundary divisor Ω_W of W

Assume $\varepsilon \in (0,1)$ and $H_W \in |6dA_W|$ general.

Define Ω_W as

$$\forall D \text{ prime divisor of } W : \quad \mu_D \Omega_W = \begin{cases} 1-\varepsilon & (D \text{ excep'l}/X) \\ 1-\varepsilon & (D \text{ component of } M_W) \\ \varepsilon & \left(\begin{array}{l} D \text{ component of } \tilde{B} \\ \text{but not of } M_W \end{array}\right) \\ 1/2 & (D = H_W) \\ 0 & (\text{otherwise}) \end{cases}$$

(※ $\tilde{B}$ = birational transform of B)

Why? To make it satisfy:

(✓) 1. (W, Ω_W) log smooth

(✓) 2. components of $\tilde{B}$, M_W have nonzero coeffs

$\Rightarrow$ any pushdown of $\Sigma_W := \operatorname{Supp} \Omega_W$ will contain the birational transform of $B+M$

(✓) 3. $\begin{cases} K_W + \Omega_W \text{ big} \\ \text{coeffs in } \Omega_W \text{ is a fixed set satisfying DCC} \end{cases}$ $\Longrightarrow$ [HMX (ACC or LCT), Lemma 7.3]

$\left(\{\varepsilon, \tfrac{1}{2}, 1-\varepsilon\}^{?}\right)$

$\exists \alpha \in (0,1)$ s.t. $K_W + \alpha \Omega_W$ big

(The fact that $K_W + \Omega_W$ is big follows from Lemma 2.46 : $K_W + \begin{pmatrix} \text{divisor in} \\ \text{dlt pair} \end{pmatrix} + \begin{pmatrix} \text{constant} \\ > 2d \end{pmatrix} \cdot (\text{nef} + \text{big Cartier divisor}) = \text{big}.$)

+. $\mathrm{vol}(K_W + \Omega_W)$ is bounded above

$\to$ will be proved in Step 3

[Step 3] $\mathrm{vol}(K_W + \Omega_W)$ is bounded above

Let $\Omega := \phi_* \Omega_W$.

$$\mathrm{vol}(K_W + \Omega_W) \leq \mathrm{vol}(K_X + \Omega) \overset{(?)}{\leq} \mathrm{vol}(K_X + B + 5dM)$$

$$< \mathrm{vol}(6dM) < (6d)^d v$$

$(\because M - (K_X + B)$ is pseudo-eff$)$ $(\because \mathrm{vol}(M) < v)$

To prove (?), it suffices to show $B + 5dM - \Omega$ is big.

$$B + 5dM - \Omega = \underbrace{\left(B + M + \tfrac{1}{2}H - \Omega\right)}_{(\geq 0)} + \left(4dM - \tfrac{1}{2}H\right)$$

$(\sim_{\mathbb{Q}} 4dM - 3dA$ which is big$)$

$= \text{big}.$ Done!

[Step 4] Construct $(\overline{X}, \Sigma_{\overline{X}})$ and show (1)

We defined $\Sigma_W := \operatorname{Supp} \Omega_W$.

To use results in [HMX (Auto)],

we first need to show $\operatorname{vol}(K_W + \Sigma_W + 2(2d+1)A_W)$

is bounded.

Recall $\exists \alpha \in (0,1)$: $K_W + \alpha \Omega_W$ is big.

So for p sufficiently large,

$$\operatorname{vol}(K_W + \Sigma_W + 2(2d+1)A_W)$$

$$\leqq \operatorname{vol}(K_W + (1 + p(1-\alpha))\Omega_W)$$

$$\leqq \operatorname{vol}(K_W + (1+p(1-\alpha))\Omega_W + p(K_W + \alpha\Omega_W))$$

$$= \operatorname{vol}((p+1)(K_W + \Omega_W)).$$ bounded!

$\Longrightarrow$ [HMX (Auto), Lemma 3.2] tells us

$$\begin{cases} \Sigma_W = \text{sum of distinct prime divisors} \\ A_W = \text{basepoint free Cartier divisor} \\ \qquad \text{s.t. } \phi_{|A_W|} \text{ birational} \end{cases}$$

then $\Sigma_W \cdot A_W^{d-1} \leq 2^d \operatorname{vol}(K_W + \Sigma_W + 2(2d+1)A_W)$.

bounded!

$\Longrightarrow$ [HMX (Acta), Lemma 2.4.2 (4)] tells us

$\exists$ constants V_1, V_2 s.t.

$\forall (W, \Omega_W)$: there exists a divisor A_W s.t.

$$\begin{cases} \mathrm{vol}(A_W) \leq V_1 \\ (\mathrm{Supp}\, \Omega_W) \cdot A_W^{d-1} \leq V_2 \\ \quad (= \Sigma_W \cdot A_W^{d-1}) \end{cases}$$

then $\underbrace{\text{the collection of } (W, \Omega_W) \text{ is bounded.}}_{\Rightarrow \text{ true!}}$

As planned, let $W \to \tilde{X}$ contraction by A_W.

Then $(\tilde{X}, \Sigma_{\tilde{X}} := (\text{pushdown of } \Sigma_W))$ is bounded,

and $\mathrm{Ex}(\tilde{X} \dashrightarrow X)$, (birational transform of $B+M$) $\subseteq \Sigma_{\tilde{X}}$.

So, $\exists$ log resolution $\overline{X} \to \tilde{X}$ of $(\tilde{X}, \Sigma_{\tilde{X}})$ s.t.

$(\Sigma_{\overline{X}} := \mathrm{Ex}(\overline{X} \to \tilde{X}) + (\text{birational transform of } \Sigma_{\tilde{X}}))$

- $(\overline{X}, \Sigma_{\overline{X}})$ log smooth
- $\mathcal{P} := \{(\overline{X}, \Sigma_{\overline{X}})\}$ bounded
- $\mathrm{Ex}(\overline{X} \dashrightarrow X)$, (birational transform of $B+M$) $\subseteq \Sigma_{\overline{X}}$.

$$\begin{tikzcd} \overline{X} \arrow[d] & W \arrow[dl] \arrow[dr] & \\ \tilde{X} \arrow[rr, dashed] & & X \end{tikzcd}$$

Done!

[Step 5] (2), (3)

(3) (✓)

(2) Take common resolution X'.

$$\begin{tikzcd} & X' \ni H_{X'} \arrow[dl] \arrow[dr] & & \\ H_{\overline{X}} \subseteq \overline{X} \arrow[dr] & & W \ni H_W \arrow[dl] \arrow[dr] & \\ & \tilde{X} \arrow[rr, dashed] & & X \end{tikzcd}$$

$H_W \sim 6dA_W$

$\Rightarrow H_{X'} \sim 6dA_{X'}$

$\Rightarrow H_{X'}, H_{\overline{X}}$ big.

$\exists\, b \in \mathbb{N}$ depending on P s.t. $\exists$ ample Cartier $C_{\overline{X}}$

s.t. $bH_{\overline{X}} - C_{\overline{X}}$ is big. $\Rightarrow bH_{X'} - C_{X'}$ is big.

($:=$ pullback of $C_{\overline{X}}$)

$$M_{\overline{X}} \cdot C_{\overline{X}}^{d-1} = M_{X'} \cdot C_{X'}^{d-1} \underset{\uparrow}{\leq} \mathrm{vol}(M_{X'} + C_{X'})$$

($\because M_{X'}, C_{X'}$ nef)

$$\leq \mathrm{vol}(M_{X'} + bH_{X'}) \leq \mathrm{vol}((1+6bd)M_{X'}).$$

↳ bounded!

$\Rightarrow$ Coeffs of $M_{\overline{X}}$ are bounded above by some fixed number c. (✓)

Done!