

MMP Learning Seminar Week 83

Tools for Effective Birationality

- singularities in bounded families
- log birational boundedness of certain pairs
- boundedness of singularities on non-klt centres

08 / 27 / 22



Prop 4.2 ("Singularities in Bounded Families")

Let $\epsilon \in \mathbb{R}^{>0}$, P bounded set of couples.

$\Rightarrow \exists \delta > 0$ depending on ϵ, P satisfying $\textcircled{\star}$

- $\textcircled{\star}$
- (X, B) ϵ -lc pair
 - T reduced divisor s.t. $(X, \text{Supp } B + T) \in P$
 - $L \geq 0$ $\mathbb{R}$ -Cartier $\mathbb{R}$ -divisor s.t.

$$\exists N \sim_{\mathbb{R}} L \quad \text{where} \quad \begin{cases} N \text{ supported on } T \\ |\text{coeffs of } N| \leq \delta \end{cases}$$

$\Rightarrow (X, B + L)$ lct.

pf.

Plan

1. Assume all members of P have $\dim d$
and induct on d

2. Reduce to the case where

- $(X, \text{Supp } B + T)$ log smooth
- B, T have no common components
- T is very ample

3. Use induction hypothesis on general member
of linear systems or components of B

[Step 1] Base case $d=1$

$$\varepsilon \leq \deg L = \deg N \leq \underbrace{\delta \deg T}_{\text{bounded}} \quad (X)$$

Assume $d \geq 2$ and assume I.H.

[Step 2-1] Restrict to log smooth case

Pick "simultaneously (!)" a log resolution $\phi: W \rightarrow X$ of $(X, B+T)$ s.t.

$$K_W + B_W = \phi^*(K_X + B) + E$$

- (W, B_W) ε -lc
- $(W, \text{Supp } B_W + T_W)$ is log smooth
& belongs to a bounded set of couple S

$$(T_W := (\text{birational transform of } T) + E_X(\phi))$$

$$\begin{cases} L_W = \phi^* L \\ N_W = \phi^* N \end{cases} \Rightarrow \exists m \in \mathbb{N} \text{ s.t. } (\text{coeffs of } N_W) \leq m\delta$$

So we can replace $P, (X, B), T, L, N, \delta$

with $S, (W, B_W), T_W, L_W, N_W, m\delta$

and assume that $(X, \text{Supp } B + T)$ is log smooth.

[Step 2-2] Reduce to the case

$\begin{cases} B, T \text{ have no common components} \\ T \text{ very ample} \end{cases} \quad (*)$

$$T = \sum_{i=1}^g T_i \quad \text{where } T_i = \text{irred components}$$

$\Rightarrow \exists$ distinct very ample prime divisors $\Lambda_1, \dots, \Lambda_{2g}$ s.t.

$\Lambda_j \notin \text{Supp}(B)$ and $T_i \sim \Lambda_{2i} - \Lambda_{2i-1}$. Let $\Lambda := \sum \Lambda_j$

$$\bullet N = \sum a_i T_i \sim N' := \sum a_i \Lambda_{2i} - \sum a_i \Lambda_{2i-1}$$

$\bullet (X, \text{Supp } B + \Lambda)$ belongs to a bounded set $\mathcal{R}$.

$\Rightarrow$ can replace T, N, P with Λ, N', R

and $(*)$ is satisfied.

[Step 3] $(X, B+L)$ is klt

Suppose $\exists D$ prime divisor on birational model of X
s.t. $a(D, X, B+L) \leq 0$.

Case 0 D is an irred divisor on X

$$T^{d-1} \cdot L = T^{d-1} \cdot N \leq \delta T^d \Rightarrow (\text{coeffs of } L) \leq \delta T^d.$$

($\because T$ very ample)

$$\Rightarrow \mu_D(B+L) \leq 1 - \varepsilon + \delta T^d < 1. \quad (*)$$

For the remaining cases, assume D is exsep² over X ,
and $V := \text{center of } D \text{ on } X$.

Case 1 $V \notin \text{Supp}(B)$

$$\text{In } a(D, X, L) = a(D, X, B+L) \leq 0.$$

$\exists r \in \mathbb{N}, H \in |rT|$:
 general member
 H irred + smooth
 $V \cap H \neq \emptyset$
 (X, H) plt
 $(X, H+L)$ not plt near any component of $V \cap H$
 (Inverse of Adjunction)
 [KM] Theorem 5.50
 $(H, L|_H)$ not plt near any component of $V \cap H$ — (#)

Also, (coeffs of $T|_H$) ≤ 2 , so $N|_H$ is supported on $T_H := \text{Supp } T|_H$ with absolute value of coeffs ≤ 2 .

Finally, $(H, T_H) \in \mathcal{Q}$ for some bounded family $\mathcal{Q}$.

$\Rightarrow$ Can we induction hypothesis on $(H, 0), T_H, L|_H, N|_H$ to define 28

$\Rightarrow (H, L|_H)$ plt. Contradiction with (#).
(X)

Case 2 V is inside some component S of B

$$\Delta := B + \underbrace{(1 - \mu_S B)}_{(=: b)} S. \quad (\dots \text{ so that } \mu_S \Delta = 1)$$

& define Δ_S s.t. $K_S + \Delta_S = (K_X + \Delta)|_S$.

$$(X, \Delta) \text{ plt} \Rightarrow (S, \Delta_S) \varepsilon\text{-lc.}$$

Since $L := \mu_S L \subseteq \delta T^d$, we can assume
(proved in Case 0)

$$L < \varepsilon \leq 1 - b \Rightarrow B + L \subseteq \Delta + L - 2S.$$

$\Rightarrow V$ is non-plt centre of $(X, \Delta + L - 2S)$

$\Rightarrow S$ ~

$$(\because \mu_S(\Delta + L - 2S) = 1)$$

$\Rightarrow (X, \Delta + L - 2S)$ is not plt near V

(Inverse of
Adjunction)

We can take $\Rightarrow (S, \Delta_S + (L - 2S)|_S)$ not plt.

$\checkmark$ $T' = T + (\text{extra components})$ so that

$\exists S' \sim S$ s.t. S' supp.-red in T' with bounded value of coeffs.

Finally, $(S, \text{Supp } \Delta_S + T'|_S) \in \mathcal{R}$ bounded set of couples.

$\Rightarrow$ use induction hypothesis on (S, Δ_S) , $T|_S$, $(L - 2S)|_S$,

$(N - 2S')|_S. \Rightarrow \text{Contradiction.}$

$\square$

Prop 4.4 ("Log birational boundedness of certain pairs")

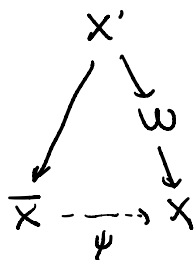
Let $d, v \in \mathbb{N}$, $\varepsilon \in \mathbb{R}^{>0}$.

$\Rightarrow \exists c \in \mathbb{R}^{>0}$ and a bounded set of couples P depending only on d, v, ε satisfying $\textcircled{*}$:

- $\textcircled{*}$ • X normal proj var with $\dim d$
- $B \geq 0$ $\mathbb{R}$ -divisor with coeffs $\in \{0\} \cup [\varepsilon, \infty)$
- $M \geq 0$ nef $\mathbb{Q}$ -divisor s.t.
 - $\left\{ \begin{array}{l} |M| \text{ defines a birational map} \\ \text{vol}(M) < v \end{array} \right.$
 - $M - (K_X + B)$ is pseudo-effective
 - $\forall D$ component of $M: \mu_D(B+M) \geq 1$

$\Rightarrow \exists$ proj log smooth couple $(\bar{X}, \Sigma_{\bar{X}}) \in \mathcal{P}$ and a birational map $\psi: \bar{X} \dashrightarrow X$ s.t.

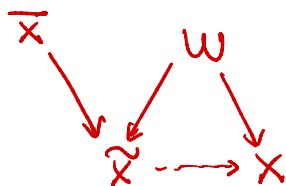
- (1) $E_X(\psi)$, (birational transform of $\text{Supp}(B+M)$) $\subseteq \text{Supp} \Sigma_{\bar{X}}$
- (2) (Coeffs of $M_{\bar{X}} := \psi^* M$) $\leq c$
- (3) $\exists$ resolution $W \rightarrow X$ s.t. $|A_W|$ is basepoint free
 $(A_W := \text{movable part of } |M_W|)$
 and $\forall$ common resolution of $\bar{X}, W$



$$A_{X'} := A_W|_{X'} \sim 0 / \bar{X}.$$

pt

Idea



(1) Since $|M|$ defines a birational map,
we can first think of taking a log resolution
 $\phi: W \rightarrow X$ of $(X, \text{Supp}(B+M))$ s.t.

$|A_W|$ is basepoint free & defines a birational contraction.

$$(M_W := \phi^*M \sim A_W + R_W)$$

(movable part)
(fixed part)

$W \rightarrow \tilde{X} =$ birational contraction determined by A_W
and then take an appropriate resolution $\overline{X} \rightarrow \tilde{X}$.

(2) To define $\Sigma_{\overline{X}}$, construct a boundary divisor on
 W s.t. satisfies boundedness conditions on volumes,
and then use the results in $[HM\chi(\text{Auro})]$
to conclude boundedness of collection of couples.

[Step 1] Construct $\phi: W \rightarrow X$, M_W, A_W as in (Idea C).

We assume A_W is a general member of $|A_W|$.

(and let $A := \phi_* A_W$)

[Step 2] Construction of boundary divisor Σ_W of W

Assume $\epsilon \in (0, 1)$ and $H_W \in |6A_W|$ general.

Define Σ_W as

$$\forall D \text{ prime divisor of } W: \mu_D \Sigma_W = \begin{cases} 1-\epsilon & (D \text{ excep' } 2/X) \\ 1-\epsilon & (D \text{ component of } M_W) \\ \epsilon & (D \text{ component of } \tilde{B} \text{ but not of } M_W) \\ 1/2 & (D = H_W) \\ 0 & (\text{otherwise}) \end{cases}$$

(* $\tilde{B}$ = birational transform of B)

Why? To make it satisfy:

(v) 1. (W, Σ_W) log smooth

(v) 2. components of $\tilde{B}$, M_W have nonzero coeffs

$\Rightarrow$ any pushdown of $\Sigma_W := \text{Supp } \Sigma_W$ will contain the birational transform of $B+M$

(v) 3. $\begin{cases} K_W + \Sigma_W \text{ big} \\ \text{coeffs in } \Sigma_W \text{ is a} \end{cases}$

$\left(\left\{ \epsilon, \frac{1}{2}, 1-\epsilon \right\} \right)^{\rightarrow}$ fixed set satisfying DCC

$\Longrightarrow$
[HMX (ACC or LCT),
Lemma 7.3]

$$\exists \alpha \in (0, 1) \text{ s.t. } K_W + \alpha \Omega_W \text{ big}$$

(The fact that $K_W + \Omega_W$ is big follows from

$$\text{Lemma 2.46 : } K_W + \left(\begin{smallmatrix} \text{divisor in} \\ \text{the pair} \end{smallmatrix} \right) + \left(\begin{smallmatrix} \text{constant} \\ \geq 2d \end{smallmatrix} \right) \cdot (\text{nef + big Cartier divisor}) = \text{big.})$$

↑ $\text{vol}(K_W + \Omega_W)$ is bounded above
 → will be proved in Step 3

[Step 3] $\text{vol}(K_W + \Omega_W)$ is bounded above

$$\text{Let } \Omega := \phi_* \Omega_W.$$

$$\text{vol}(K_W + \Omega_W) \stackrel{(?)}{\leq} \text{vol}(K_X + \Omega) \leq \text{vol}(K_X + B + 5dM)$$

$$< \text{vol}(6dM) < (6d)^d v$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ (\because M - (K_X + B) \text{ is pseudo-eff}) & & (\because \text{vol}(M) < v) \end{array}$$

To prove (?), it suffices to show $B + 5dM - \Omega$ is big.

$$\begin{aligned} B + 5dM - \Omega &= \underbrace{\left(B + M + \frac{1}{2}H - \Omega \right)}_{(\geq 0)} + \left(4dM - \frac{1}{2}H \right) \\ &= \text{big.} \quad \text{Done!} \end{aligned}$$

($\sim_{\mathbb{Q}} 4dM - 3dA$
which is big)

[Step 4] Construct $(\bar{X}, \Sigma_{\bar{X}})$ and show (1)

We defined $\Sigma_W := \text{Supp } \Omega_W$.

To use results in $[HMX(\text{Auto})]$,

we first need to show $\text{vol}(K_W + \Sigma_W + 2(2d+1)A_W)$ is bounded.

(Recall) $\exists \alpha \in (0, 1) : K_W + \alpha \Sigma_W$ is big.

So for p sufficiently large,

$$\text{vol}(K_W + \Sigma_W + 2(2d+1)A_W)$$

$$\leq \text{vol}(K_W + (1+p(1-\alpha))\Sigma_W)$$

$$\leq \text{vol}(K_W + (1+p(1-\alpha))\Sigma_W + p(K_W + \alpha\Sigma_W))$$

$$= \text{vol}((p+1)(K_W + \Sigma_W)). \rightarrow \text{bounded!}$$

$\Rightarrow [HMX(\text{Auto}), \text{ Lemma 2.2}]$ tells us

$\left\{ \begin{array}{l} \Sigma_W = \text{sum of disjoint prime divisors} \\ A_W = \text{basepoint free Cartier divisor} \end{array} \right.$

s.e. $\phi_{|A_W|}$ birational

$$\text{then } \Sigma_W \cdot A_W^{d-1} \leq 2^d \text{vol}(K_W + \Sigma_W + 2(2d+1)A_W).$$

$\hookrightarrow$ bounded!

$\Rightarrow [HMX(A_{\text{geo}}), \text{ Lemma 2.4.2 (4)}]$ tells us

$\exists$ constants V_1, V_2 s.t.

$\forall (W, \Sigma_W)$: there exists a divisor A_W s.t.

$$\begin{cases} \text{vol}(A_W) \leq V_1 \\ (\text{Supp } \Sigma_W) \cdot A_W^{\text{div}} \leq V_2 \\ (= \Sigma_W \cdot A_W^{\text{div}}) \end{cases}$$

then the collection of (W, Σ_W) is bounded.

$\Rightarrow$ true!

As planned, let $W \rightarrow \tilde{X}$ contraction by A_W .

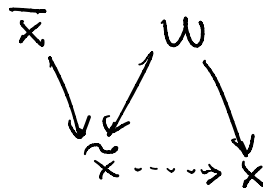
Then $(\tilde{X}, \Sigma_{\tilde{X}} := (\text{pushdown of } \Sigma_W))$ is bounded,

and $\text{Ex}(\tilde{X} \dashrightarrow X)$, (birational transform of $B+M$) $\subseteq \Sigma_{\tilde{X}}$.

So, $\exists$ log resolution $\bar{X} \rightarrow \tilde{X}$ of $(\tilde{X}, \Sigma_{\tilde{X}})$ s.t.

$$(\Sigma_{\bar{X}} := \text{Ex}(\bar{X} \rightarrow \tilde{X}) + (\text{birational transform of } \Sigma_{\tilde{X}}))$$

- $(\bar{X}, \Sigma_{\bar{X}})$ log smooth
- $\mathcal{P} := \{(\bar{X}, \Sigma_{\bar{X}})\}$ bounded
- $\text{Ex}(\bar{X} \dashrightarrow X)$, (birational transform of $B+M$) $\subseteq \Sigma_{\bar{X}}$.

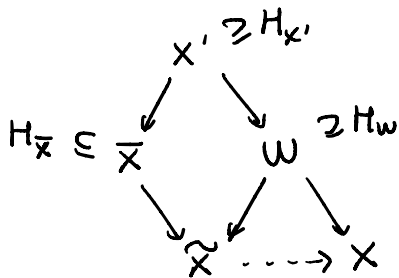


Done!

[Step 5] (2), (3)

(3) (✓)

(2) Take common resolution X' .



$$H_W \sim b d A_W$$

$$\Rightarrow H_{X'} \sim b d A_{X'}$$

$$\Rightarrow H_{X'}, H_{\bar{X}} \text{ big.}$$

$\exists b \in \mathbb{N}$ depending on P s.t. $\exists$ ample Cartier $C_{\bar{X}}$

s.t. $b H_{\bar{X}} - C_{\bar{X}}$ is big. $\Rightarrow b H_{X'} - C_{X'}$ is big.
 (': pullback of $C_{\bar{X}}$)

$$M_{\bar{X}} \cdot C_{\bar{X}}^{d-1} = M_{X'} \cdot C_{X'}^{d-1} \leq \text{vol}(M_{X'} + C_{X'})$$

↑
($\because M_{X'}, C_{X'}$ nef)

$$\leq \text{vol}(M_{X'} + b H_{X'}) \leq \text{vol}((1 + b d) M_{X'}).$$

$\hookrightarrow$ bounded!

$\Rightarrow$ Coeffs of $M_{\bar{X}}$ are bounded above by some fixed number c . (✓)

Done!